"Kernel methods in machine learning"
Homework 1
Due on January 19, 2022, 3pm

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Exercice 1. Kernels

Study whether the following kernels are positive definite:

1. $\mathcal{X} = \mathbb{R}_+, \quad K(x, x') = \min(x, x')$
2. $\mathcal{X} = \mathbb{R}_+, \quad K(x, x') = \max(x, x')$
3. Let $\mathcal{X}$ be a set and $f, g : \mathcal{X} \to \mathbb{R}_+$ two non-negative functions:

$$\forall x, y \in \mathcal{X} \quad K(x, y) = \min(f(x)g(y), f(y)g(x))$$

Exercice 2. Non-expansiveness of the Gaussian kernel

Consider the Gaussian kernel $K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ such that for all pair of points x, x' in $\mathbb{R}^p$,

$$K(x, x') = e^{-\frac{\alpha}{2}\|x-x'\|^2},$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^p$. Call $\mathcal{H}$ the RKHS of K and consider its RKHS mapping $\varphi : \mathbb{R}^p \to \mathcal{H}$ such that $K(x, x') = \langle\varphi(x), \varphi(x')\rangle_{\mathcal{H}}$ for all x, x' in $\mathbb{R}^p$. Show that

$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} \leq \sqrt{\alpha}\|x - x'\|.$$

The mapping is called non-expansive whenever $\alpha \leq 1$.

Exercice 3. Uniqueness of the RKHS

Prove that if $K : \mathcal{X} \times \mathcal{X}$ is a positive definite function, then it is the r.k. of a unique RKHS. (Hint: consider the linear space spanned by the functions $K_x : t \mapsto K(x,t)$, and use the fact that a linear subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$ is dense in $\mathcal{H}$ if and only 0 is the only vector orthgonal to all vectors in $\mathcal{F}$)