"Kernel methods in machine learning" Homework 4 Due March 9th, 2021, 1:30pm

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Exercice 1. B_n-splines

The convolution between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f \star g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du,$$

when this integral exists.

Let now the function:

$$I(x) = \begin{cases} 1 & \text{si } -1 \leq x \leq 1, \\ 0 & \text{si } x < -1 \text{ ou } x > 1, \end{cases}$$

and $B_n = I^{\star n}$ for $n \in \mathbb{N}_*$ (that is, the function I convolved n times with itself: $B_1 = I, B_2 = I \star I, B_3 = I \star I \star I$, etc...).

Is the function $k(x, y) = B_n(x - y)$ a positive definite kernel over $\mathbb{R} \times \mathbb{R}$? If yes, describe the corresponding reproducing kernel Hilbert space.

Exercice 2. Diffusion kernel on a grid

Let $0 = \lambda_1 \leq \ldots \leq \lambda_n \in \mathbb{R}$ be the eigenvalues and $e_1, \ldots, e_n \in \mathbb{R}^n$ the eigenvectors of the Laplacian L_1 of the line graph with n vertices[1].

1. Show that the eigenvalues of the Laplacian L_2 of the $n \times n$ square grid[2] are $\lambda_{ij} = \lambda_i + \lambda_j$ for $i, j = 1, \ldots, n$, and compute the corresponding eigenvectors $e_{ij} \in \mathbb{R}^{n^2}$ as a function of e_i and e_j.

[1] Vertices $V_1 = \{1, \ldots, n\}$, edges $E_1 = \{(i, j) \in V_1 \times V_1 \text{ such that } |i - j| = 1\}$.

[2] Vertices $V_2 = V_1 \times V_1$, edges $E_2 = \{((i, j), (i', j')) \in V_2 \times V_2 \text{ such that } |i - i'| + |j - j'| = 1\}$

2. Let $K_1 = e^{-tL_1} \in \mathbb{R}^{n\times n}$ and $K_2 = e^{-tL_2} \in \mathbb{R}^{n^2\times n^2}$ be diffusion kernels, respectively on the line graph and on the square grid. Show that, for any $i, j, k, l \in \{1, \ldots, n\}$,

$$K_2\left((i,j),(k,l)\right) = K_1(i,k)K_1(j,l)\,.$$

3. Assuming the complexity of computing the exponential of an $n\times n$ matrix is $O(n^3)$, what is the complexity of computing K_1? Of computing K_2?