Let K p.d. and $\gamma > 0$.

$\boxed{\text{RKHS of } \gamma K \text{ ?}}$

Then $\tilde{K} = \gamma K$ is p.d because: 1) symmetry

2) $\sum_{i,j}\sum a_i a_j \tilde{K}(x_i, x_j) = \underbrace{\gamma}_{>0} \underbrace{\sum_{i,j}\sum a_i a_j K(x_i, x_j)}_{\geq 0}$

What is the RKHS?

$\boxed{\text{Analysis}}$

$\forall n \in \mathbb{N}$, $(\tilde{a}_1, \dots \tilde{a}_n) \in \mathbb{R}^n$, $(x_1, \dots, x_n) \in \mathcal{X}^n$,

$$\tilde{f} = \tilde{\sum_{i=1}} \tilde{a}_i \tilde{K}_{x_i} \in \tilde{H} = \hat{\sum_{i=1}} \underbrace{\gamma \tilde{a}_i}_{a_i} K_{x_i} \in H \qquad H = \tilde{H} \text{ ?}$$

$$\|\tilde{f}\|^2_{\tilde{H}} = \tilde{\sum_{i,j=1}\sum} \tilde{a}_i \tilde{a}_j \hat{K}(x_i, x_j)$$

$$= \sum_{i,j}\sum \frac{a_i}{\gamma} \frac{a_j}{\gamma} \times \gamma K(x_i, x_j) = \frac{1}{\gamma} \|\tilde{f}\|^2_H$$

$\boxed{\text{Candidate: } \tilde{H} = H, \ \|\cdot\|_{\tilde{H}} = \frac{1}{\sqrt{\gamma}} \|\cdot\|_H}$

$\boxed{\text{Synthesis}}$

1) This is a Hilbert space (through isomorphic mapping $\begin{cases} H \longrightarrow H \\ f \longmapsto \sqrt{\gamma}\, f \end{cases}$)

2) $\forall x \in \mathcal{X}, \quad \tilde{K}_x = \gamma K_x \in H = \tilde{H}$

3) $\forall \tilde{f} \in \tilde{H}, \forall x \in \mathcal{X},$

$$\langle \tilde{f}, \tilde{K}_x \rangle_{\tilde{H}} = \frac{1}{\gamma} \langle \tilde{f}, \tilde{K}_x \rangle_H = \frac{1}{\gamma} \langle \tilde{f}, \gamma K_x \rangle_H = \frac{\gamma}{\gamma} \langle \tilde{f}, K_x \rangle_H = \tilde{f}(x)$$

$\square$

$\boxed{\text{RKHS of } k_1 + k_2 \text{ ?}}$ Let $(H_i, \langle\cdot,\cdot\rangle_{H_i})$ RKHS of k_i

Analysis: $$f = \sum_{i=1}^{n} a_i (k_{1x_i} + k_{2x_i}) = \underbrace{\sum_{i=1}^{n} a_i k_{1x_i}}_{f_1 \in H_1} + \underbrace{\sum_{i=1}^{n} a_i k_{2x_i}}_{f_2 \in H_2}$$

$$\in H_1 + H_2$$

and $$\|f\|^2 = \sum_{i,j=1}^{n} \sum a_i a_j (k_1(x_i, x_j) + k_2(x_i, x_j)).$$

$$= \sum_{i,j} \sum a_i a_j k_1(x_i, x_j) + \sum_{i,j} \sum a_i a_j k_2(x_i, x_j)$$

$$= \|f_1\|^2_{H_1} + \|f_2\|^2_{H_2}$$

Candidate: $\{ f = f_1 + f_2 \in H_1 + H_2 ,\ \|f\|^2 = \|f_1\|^2_{H_1} + \|f_2\|^2_{H_2} \}$

Problem: Decomposition may not be unique!

How to make $H_1 + H_2$ a Hilbert space?

Let $s : H_1 \times H_2 \longrightarrow H_1 + H_2$

$(f_1, f_2) \longmapsto f_1 + f_2$

Hilbert space with $\|(f_1, f_2)\|^2_{H_1 \times H_2} = \|f_1\|^2_{H_1} + \|f_2\|^2_{H_2}$

s linear, surjective, but not necessarily injective.

Let $N = s^{-1}(\{0\}) = \{(f_1, f_2) \in H_1 \times H_2 : f_1 + f_2 = 0\}$

$= \{(f, -f) : f \in H_1 \cap H_2\}$

Then: • N is closed in $(H_1 \times H_2, \langle \cdot,\cdot \rangle_{H_1 \times H_2})$

(take a sequence $(f_n, g_n) \in N \xrightarrow[n\to\infty]{} (f,g)$)

then $f_n \xrightarrow{H_1} f$, $g_n \xrightarrow{H_2} g$ $\Rightarrow \begin{cases} f(x) = \lim\limits_{n\to\infty} f_n(x) \\ g(x) = \lim\limits_{n\to\infty} g_n(x) \\ \forall n \; f_n(x) - g_n(x) = 0 \end{cases} \Rightarrow f = g \Downarrow (f,g) \in N$

• By projection theorem, $\boxed{H_1 \times H_2 = N \oplus N^\perp}$

Then $\tilde{S} = S/_{N^\perp}$ is an isomorphism $\boxed{N^\perp \overset{\hat{S}}{\simeq} H_1 + H_2}$

$\Rightarrow$ $H_1 + H_2$ Hibert space with $\boxed{\|f\|_{H_1+H_2} = \|\tilde{S}^{-1}(f)\|_{H_1 \times H_2}}$

It is the RKHS because:

1) $\forall x, \quad K_x = K_{1x} + K_{2x} \in H_1 + H_2$

2) $\forall x \in X \quad \forall f \qquad \in H_1 + H_2$

$A + B = K_x$

let $\tilde{S}^{-1}(f) = (f_1, f_2) \qquad \tilde{S}^{-1}(K_x) = (A, B)$

$f_1 + f_2 = f$

then $\langle f, K_x \rangle_{H_1 + H_2} = \langle (f_1, f_2), (A, B) \rangle_{H_1 \times H_2}$

$$= \langle (f_1, f_2), (K_{1x}, K_{2x}) \rangle_{H_1 \times H_2} + \langle \underbrace{(f_1, f_2)}_{\in N^\perp}, \underbrace{(A, B) - (K_{1x}, K_{2x})}_{\in N} \rangle_{H_1 \times H_2}$$

$$= \langle f_1, K_{1x} \rangle_{H_1} + \langle f_2, K_{2x} \rangle_{H_2} + 0$$

$$= f_1(x) + f_2(x) = f(x) \qquad \square$$

Furthermore, $\forall (f_1, f_2) \in H_1 \times H_2$

$$\|f_1\|^2_{H_1} + \|f_2\|^2_{H_2} = \|(f_1, f_2)\|^2_{H_1 \times H_2}$$

$$= \|\underbrace{(f_1^N, f_2^N)}_{N}\|^2_{H_1 \times H_2} + \|\underbrace{\tilde{S}^{-1}(f_1 + f_2)}_{N^\perp}\|^2_{H_1 \times H_2}$$

$$\geq \|f_1 + f_2\|^2_{H_1 + H_2}$$

w. equality iff $(f_1^N, f_2^N) = (0,0)$ which is always possible

because $(f_1^N, f_2^N) \in N \Leftrightarrow f_1^N = g,\ f_2^N = -g,\ g \in H_1 \cap H_2$

$$\Rightarrow \quad f_1 = f_1^\perp + g \quad f_2 = f_2^\perp - g \Longrightarrow \text{take } f = f_1^\perp + f_2^\perp \quad \square$$

Conclusion : The RKHS of $K_1 + K_2$ is $H_1 + H_2$, endowed with the norm :

$$\boxed{\forall f \in H_1 + H_2, \quad \|f\|^2 = \min_{\substack{f_1 \in H_1 \\ f_2 \in H_2 \\ f_1 + f_2 = f}} \left\{ \|f_1\|_{H_1}^2 + \|f_2\|_{H_2}^2 \right\}}$$