"Kernel methods in machine learning"
Homework 3
Due on March 15, 2023, 3pm

March 1, 2023

Exercice 1. B_n-splines
The convolution between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f \star g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du,$$

when this integral exists.

Let now the function:

$$I(x) = \begin{cases} 1 & \text{si } -1 \leq x \leq 1, \\ 0 & \text{si } x < -1 \text{ ou } x > 1, \end{cases}$$

and $B_n = I^{\star n}$ for $n \in \mathbb{N}_*$ (that is, the function I convolved n times with itself: $B_1 = I, B_2 = I \star I, B_3 = I \star I \star I$, etc...).

Is the function $k(x, y) = B_n(x - y)$ a positive definite kernel over $\mathbb{R} \times \mathbb{R}$? If yes, describe the corresponding reproducing kernel Hilbert space.

Exercice 2. Sobolev spaces

1. Let

$$\mathcal{H} = \left\{ f : [0,1] \to \mathbb{R}\,,\ \text{absolutely continuous}, f' \in L^2([0,1]), f(0) = 0 \right\}\ ,$$

endowed with the bilinear form

$$\forall f, g \in \mathcal{H}\,, \quad \langle f, g\rangle_{\mathcal{H}} = \int_0^1 f'(u)g'(u)du\,.$$

Show that $\mathcal{H}$ is an RKHS, and compute its reproducing kernel.

2. Same question when

$$\mathcal{H} = \left\{f : [0,1] \to \mathbb{R}\,,\ \text{absolutely continuous}, f' \in L^2([0,1]), f(0) = f(1) = 0\right\}\,,$$

3. Same question, when $\mathcal{H}$ is endowed with the bilinear form:

$$\forall f, g \in \mathcal{H}\,, \quad \langle f, g\rangle_{\mathcal{H}} = \int_0^1 \left(f(u)g(u) + f'(u)g'(u)\right) du\,.$$

4. Same question when

$$\mathcal{H} = \left\{f : [0,1] \to \mathbb{R}\,,\ f' \text{ exists and absolutely continuous}, f'' \in L^2([0,1]), f(0) = f('0) = 0\right\}\,,$$

endowed with the bilinear form

$$\forall f, g \in \mathcal{H}\,, \quad \langle f, g\rangle_{\mathcal{H}} = \int_0^1 f''(u)g''(u)du\,.$$