"Kernel methods in machine learning"
Homework 1
Due on January 29, 2025, 1:30 pm

Michael Arbel, Julien Mairal

Exercice 1. Kernels

Study whether the following kernels are positive definite:

1. $\mathcal{X} = \mathbb{R}_+, \quad K(x, x') = \min(x, x')$
2. $\mathcal{X} = \mathbb{R}_+, \quad K(x, x') = \max(x, x')$
3. Let $\mathcal{X}$ be a set and $f, g : \mathcal{X} \to \mathbb{R}_+$ two non-negative functions:
$$\forall x, y \in \mathcal{X} \quad K(x, y) = \min(f(x)g(y), f(y)g(x))$$

Exercice 2. Non-expansiveness of the Gaussian kernel

Consider the Gaussian kernel $K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ such that for all pair of points x, x' in $\mathbb{R}^p$,
$$K(x, x') = e^{-\frac{\alpha}{2}\|x-x'\|^2},$$
where $\|.\|$ is the Euclidean norm on $\mathbb{R}^p$. Call $\mathcal{H}$ the RKHS of K and consider its RKHS mapping $\varphi : \mathbb{R}^p \to \mathcal{H}$ such that $K(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$ for all x, x' in $\mathbb{R}^p$. Show that
$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} \leq \sqrt{\alpha}\|x - x'\|.$$
The mapping is called non-expansive whenever $\alpha \leq 1$.

Exercice 3. RKHS

1. Let K_1 and K_2 be two positive definite kernels on a set $\mathcal{X}$, and α, β two positive scalars. Show that $\alpha K_1 + \beta K_2$ is positive definite, and describe its RKHS.

2. Prove that for any p.d. kernel K on a space $\mathcal{X}$, a function $f : \mathcal{X} \to \mathbb{R}$ belongs to the RKHS $\mathcal{H}$ with kernel K if and only if there exists $\lambda > 0$ such that $K(\mathbf{x}, \mathbf{x}') - \lambda f(\mathbf{x}) f(\mathbf{x}')$ is p.d.